第五课：球坐标系

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1. 角动量$L=\vec{r}\times\vec{p}$;
2. $\begin{cases} L=\vec{r}\times i\hbar\nabla=i\hbar\vec{r}\times\nabla \\ \vec{\mathrm{r}}=\mathrm{xi}+\mathrm{yj}+\mathrm{zk} \\ \nabla=\frac{\partial}{\partial \mathrm{x}}\mathrm{i}+\frac{\partial}{\partial \mathrm{y}}\mathrm{j}+\frac{\partial}{\partial \mathrm{z}}\mathrm{k} \end{cases}$;
3. $\Rightarrow L=i\hbar\vec{r}\times\nabla=i\hbar\begin{vmatrix} i & j & k \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}=i\hbar\left[\left(y\cdot\frac{\partial}{\partial z}-z\cdot\frac{\partial}{\partial y}\right)i+\left(z\cdot\frac{\partial}{\partial x}-x\cdot\frac{\partial}{\partial z}\right)j+\left(x\cdot\frac{\partial}{\partial y}-y\cdot\frac{\partial}{\partial x}\right)k\right]$;
4. 令:$L=L_x i+L_y j+L_z k$;
5. $\begin{cases} x=r\cdot\sin\theta\cdot\cos\phi \\ y=r\cdot\sin\theta\cdot\sin\phi \\ z=r\cdot\cos\theta \end{cases}\Rightarrow\begin{cases} r^2=x^2+y^2+z^2 \\ \cos\theta=\frac{z}{r} \\ \frac{y}{x}=tg\ \phi \end{cases}$;
6. $\frac{\partial}{\partial x}=\frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial x}$;
7. $\because r^2=x^2+y^2+z^2\Rightarrow\frac{dr}{dx}\cdot 2r=2x\Rightarrow\frac{dr}{dx}=\frac{x}{r}$;
8. $\because x=r\cdot\sin\theta\cdot\cos\phi\Rightarrow\frac{dr}{dx}=\sin\theta\cdot\cos\phi$;
9. $\because r\cdot\cos\theta=z\Rightarrow\frac{dr}{dx}\cdot\cos\theta+r\cdot(-\sin\theta)\cdot\frac{d\theta}{dx}=0\Rightarrow\frac{d\theta}{dx}=\frac{\cos\phi\cdot\cos\theta}{r}$;
10. $\because y=x\cdot tg\ \phi\Rightarrow\theta=tg\ \phi+x\cdot\sec\phi\cdot\frac{d\phi}{dx}\Rightarrow\frac{d\phi}{dx}=-\frac{\sin\phi}{\cos\phi}\cdot\frac{1}{x}\cdot\cos\phi\cdot\cos\phi=-\frac{\sin\phi\cdot\cos\phi}{r\cdot\sin\theta\cdot\cos\phi}=-\frac{1}{r}\cdot\frac{\sin\phi}{\sin\theta}$;
11. $\Rightarrow\frac{\partial}{\partial x}=\sin\theta\cdot\cos\phi\cdot\frac{\partial}{\partial r}+\frac{\cos\theta\cdot\cos\phi}{r}\cdot\frac{\partial}{\partial\theta}+\left(-\frac{1}{r}\right)\cdot\frac{sin\phi}{sin\theta}\cdot\frac{\partial}{\partial\phi}$;
12. 同理：$\begin{cases} \frac{\partial}{\partial x}=\sin\theta\cdot\cos\phi\cdot\frac{\partial}{\partial r}+\frac{\cos\theta\cdot\cos\phi}{r}\cdot\frac{\partial}{\partial\theta}+\left(-\frac{1}{r}\right)\cdot\frac{sin\phi}{sin\theta}\cdot\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial y}=sin\theta\cdot sin\phi\cdot\frac{\partial}{\partial r}+\frac{1}{r}cos\theta\cdot sin\phi\cdot\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{cos\phi}{sin\theta}\cdot\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial z}=cos\theta\cdot\frac{\partial}{\partial r}+\left(-\frac{1}{r}\right)sin\theta\cdot\frac{\partial}{\partial\theta}+0 \end{cases}$;
13. $\because L^2=L_x^2+L_y^2+L_z^2$;
14. $\Rightarrow L^2=-\hbar^2[\left(y\cdot\frac{\partial}{\partial z}-z\cdot\frac{\partial}{\partial y}\right)^2+\left(z\cdot\frac{\partial}{\partial x}-x\cdot\frac{\partial}{\partial z}\right)^2+\left(x\cdot\frac{\partial}{\partial y}-y\cdot\frac{\partial}{\partial x}\right)^2]$;
15. $\Rightarrow L^2=-\hbar^2\left[\frac{1}{sin\theta}\cdot\frac{\partial}{\partial\theta}\left(sin\theta\cdot\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}\right]$;
16. 本征方程：算符作用在波函数上等于一个确定的值乘波函数;
17. $\widehat{L_z}=-i\hbar\cdot\frac{\partial}{\partial\phi}\Rightarrow\widehat{L_z}\psi=L_z\cdot\psi\Rightarrow-i\hbar\cdot\frac{\partial}{\partial\phi}\psi=L_z\psi$($\mathrm{L_z}$为实数，可观测);
18. $\Rightarrow\frac{\partial\psi}{\partial\phi}=1=-\frac{1}{i\hbar}\cdot L_z\cdot\psi=\frac{i}{\hbar}\cdot\mathrm{L}_z\cdot\psi$;
19. $\psi=ce^{\frac{iL_z}{\hbar}\phi}$;
20. 由角动量边界条件：$\psi(\phi)=\psi(\phi+2\pi)$;
21. $ce^{\frac{iL_z}{\hbar}\phi}=ce^{\frac{iL_z}{\hbar}(\phi+2\pi)}$;
22. $\because e^{\frac{iL_z2\pi}{\hbar}}=\cos\left(\frac{L_z2\pi}{\hbar}\right)+isin\left(\frac{L_z2\pi}{\hbar}\right)=1\ and\ \frac{L_z2\pi}{\hbar}=2\pi m\Rightarrow L_z=m\hbar$;
23. $\because\int_0^{2\pi}|\psi|^2d\psi=1\Rightarrow\int_0^{2\pi}c^2\left|e^{im\phi}\right|^2d\psi=1\Rightarrow c^2\int_0^{2\pi}1d\psi=1\Rightarrow c=\sqrt{\frac{1}{2\pi}}$;
24. $\begin{cases} L_z=m\hbar \\ \psi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi} \end{cases}$;
25. $\widehat{L_2}Y(\theta,\phi)=\lambda\hbar^2Y(\theta,\phi)$;
26. $-\hbar^2\left[\frac{1}{sin\theta}\cdot\frac{\partial}{\partial\theta}\left(sin\theta\cdot\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}\right]Y(\theta,\phi)=\lambda\hbar^2Y(\theta,\phi)$;

26. $-\hbar^2\left[\frac{1}{sin\theta}\cdot\frac{\partial}{\partial\theta}\left(sin\theta\cdot\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}\right]Y(\theta,\phi)=\lambda\hbar^2Y(\theta,\phi)$;
27. $\widehat{L_z^2}=-i\hbar\cdot\frac{\partial}{\partial\phi}\Rightarrow L_z^2=-\hbar^2\cdot\frac{\partial^2}{\partial\phi^2}$;
28. 式$-\hbar^2\left[\frac{1}{sin\theta}\cdot\frac{\partial}{\partial\theta}\left(sin\theta\cdot\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}\right]Y(\theta,\phi)=\lambda\hbar^2Y(\theta,\phi)$中，前半部分是关于$\theta$的函数，后半部分关于$\phi$，因此可以分离变量;
29. $\because\widehat{L_z}\phi(\phi)=m\hbar\phi(\phi)$;
30. $Y(\theta,\phi)=\theta(\theta)\cdot\phi(\phi)\Rightarrow L_z^2\phi(\phi)=m^2\hbar^2\phi(\phi)$;
31. 消掉ϕ $\Rightarrow\left[-\frac{\hbar^2}{sin\theta}\cdot\frac{\partial}{\partial\theta}\left(sin\theta\cdot\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\cdot m^2\hbar^2\right]Y(\theta,\phi)=\lambda\hbar^2Y(\theta,\phi)$;
32. 解得：$\frac{1}{sin\theta}\cdot\frac{\partial}{\partial\theta}\left(sin\theta\cdot\frac{\partial}{\partial\theta}\right)+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\theta=0$;
33. $Y(\theta,\phi)=\begin{cases}(-1)^mN_{lm}\cdot P_l^m(cos\theta)e^{im\phi}, m=0\\(-1)^mY_{1-m}^*(\theta,\phi), m=-1\end{cases}, N_{lm}=\sqrt{\frac{(1-|m|)!(2l+1)}{4\pi(1+|m|)!}}$;
34. l表示角量子数，m表示磁量子数;
35. 当l确定时，m可取$0,\pm1,\pm2\ldots,\pm l$,此时简并度为2l+1;