第六课：再谈算符

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1. 算符作用在波函数上等于一个确定的值乘波函数：$\hat{F}\psi = \mathrm{f}\cdot\psi$；
2. 厄米算符的性质：
   1. 厄米算符的平均值时实数：
      $\bar{F} = \int d\tau\psi^*\hat{F}\psi = \int d\tau(\hat{F}\psi) * \psi = \int d\tau\left(\psi^*(\hat{F}\psi)\right)^* = \bar{F}^*$
      ∵ $\bar{F} = \bar{F}^*$ ⇒ F̄是实数
   2. 平均值都是实数的算符必定是厄米算符(1的逆定理)：
      $\bar{F} = \bar{F}^*$
      左侧: $\bar{F} = \int \psi^*\hat{F}\psi d\tau = \int d\tau(\bar{F}\psi)^*\psi$
      取$\psi = \psi_1 + \mathrm{c}\psi_2$代入F̄ $= \int d\tau\psi^*\hat{F}\phi = \int d\tau(\psi_1 + c\psi_2)^*\hat{F}(\psi_1 + c\psi_2)$
      $\int d\tau\psi_1^*\hat{F}\psi_1 + |c|^2\int d\tau\psi_2^*\hat{F}\psi_2 + c\int d\tau\psi_1^2\hat{F}\psi_2 + \int d\tau c\psi_2^*\hat{F}\psi_1$
      $\int d\tau(\hat{F}\psi)^*\psi = \int d\tau\left(\hat{F}(\psi_1 + c\psi_2)\right)^*(\psi_1 + c\psi_2)$
      $= \int d\tau(\hat{F}\psi_1)^*\psi_1 + \int(\hat{F}\psi_1)^*c\psi_2 + \int d\tau(\hat{F}c\psi_2)^*\psi_1 + \int d\tau(\hat{F}c\psi_2)c\psi_2$
      $\int d\tau(\hat{F}\psi_1)^*\psi_1 = \left(\int d\tau\psi_1^*(\hat{F}\psi_1)\right)^*$
      $\int d\tau(\hat{F}\psi_2)^*\psi_2 = \left[\int d\tau\psi_2^*(F\psi_2)\right]^*$
      ⇒ $\bar{F} = \bar{F}^*$
      ∵ 左式 = 右式
      ⇒ $c\left(\int d\tau\psi_1^*\hat{F}\psi_2 - \int d\tau(F\psi_1)^*\psi_2\right) = c^*\left(\int d\tau(\hat{F}\psi_2)^*\psi_1 - \int d\tau\psi_2^*\hat{F}\psi_1\right)$
      令c=1:$\int d\tau\psi_1^*\hat{F}\psi_2 - \int d\tau(F\psi_1)^*\psi_2 = \int d\tau(\hat{F}\psi_2)^*\psi_1 - \int d\tau\psi_2^*\hat{F}\psi_1$
      令c=i:$\int d\tau\psi_1^*\hat{F}\psi_2 - \int d\tau(F\psi_1)^*\psi_2 = -\left(\int d\tau(\hat{F}\psi_2)^*\psi_1 - \int d\tau\psi_2^*\hat{F}\psi_1\right)$
      $\begin{cases}\int d\tau\psi_1^*\hat{F}\psi_2 = \int d\tau(F\psi_1)^*\psi_2 \\ \int d\tau\psi_2^*\hat{F}\psi_1 = \int d\tau(F\psi_2)^*\psi_1\end{cases}$(厄米算符表达式)
   3. 本征方程：$\hat{F}\psi = \mathrm{f}\cdot\psi$；
      涨落：$\overline{|\Delta F|^2} = \overline{\left(\hat{F} - \bar{F}\right)^2} = \int\psi^*\left(\hat{F} - \bar{F}\right)\psi d\tau$，其中：F̂是厄米算符,F̄是实数,
      其平均值是实数；
      $\bar{F}^2 = \int d\tau\psi^*F^2\psi = \int d\tau\psi^*\hat{F}\hat{F}\psi d\tau = \int d\tau(\hat{F}\psi)^*(\hat{F}\psi) = \int d\tau\left|\hat{\mathrm{F}}\psi\right|^2 \geq 0$
      把F换成ΔF：
      $\overline{|\Delta F|^2} = \int\left|\hat{F} - \bar{F}\psi\right|^2 \geq 0$
      当$\overline{|\Delta F|^2} = 0$时，测量涨落最小，即$\left|\hat{F} - \bar{F}\right|\psi = 0$，$\hat{F}\psi = \bar{\mathrm{F}}\psi$是本征方程；
   4. 正交归一性：厄米算符属于不同本征值的本征函数正交；
      $\hat{F}\phi_n = F_n\phi_n, \hat{F}\phi_m = F_m\phi_m$，证明它们正交。
      取 $\hat{F}\phi_m = F_m\phi_m$的共轭$\left(\hat{F}\phi_m\right)^* = (F_m\phi_m)^* = F_m\cdot\phi_m^*$
      右乘ϕ_n并积分：$\int\left(\hat{F}\phi_m\right)^*\phi_n d\tau = F_m\cdot\int\phi_m^*\phi_n d\tau$
      由厄米算符的性质：$\int\left(\hat{F}\phi_m\right)^*\phi_n d\tau = \int\phi_m^*\hat{F}\phi_n d\tau = \int\phi_m^*F_n\phi_n d\tau = F_n\int\phi_m^*\phi_n d\tau = F_m\cdot\int\phi_m^*\phi_n d\tau$
      ∵ $\forall F_m \neq F_n \Rightarrow \int\phi_m^*\phi_n d\tau = 0 (m \neq n)$(正交性)

$\int \phi_m^* \phi_n d\tau = 1(m=n)$(归一性)

正交归一性$\int \phi_m^* \phi_n d\tau = \theta_{mn}$

5. 完备性：$\hat{F}\phi_n = \lambda_n \phi_n (n \in N^*)$

λ_n: $\lambda_1, \lambda_2, \lambda_3 \ldots$(本征值)

ϕ_n: $\phi_1, \phi_2, \phi_2 \ldots$(本征函数)

对于厄米算符$\hat{F}$:

本征函数任意$\psi(x)$可以按$\phi_n(x)$展开：$\psi(x) = \Sigma_n c_n \phi_n(x)$

左乘ϕ_m^*并积分：$\int \phi_m^* \psi(x) = \int \phi_m^* \Sigma_n c_n \phi_n d\tau$

$\Rightarrow \int \phi_m^* \psi(x) d\tau = \Sigma_n c_n \int \phi_m^* \phi_n d\tau = \Sigma_n c_n \cdot \theta_{mn} = c_m$

6. 当$\psi(x)$归一化时，系数c_n也是归一化的：

$1 = \int \psi^*(x)\psi(x)d\tau, \quad \psi(x) = \Sigma_n c_n \phi_n$

$\Rightarrow 1 = \int (\Sigma_n C_N \phi_n)^* \cdot (\Sigma_m c_m \phi_m) d\tau = \Sigma_n \Sigma_m c_n c_m \int \phi_n^* \phi_m d\tau = \Sigma_n \Sigma_m c_n c_m \theta_{mn}$

$= \Sigma_n |c_n|^2 = 1$

7. 力学量的平均值：$\bar{F} = \int \psi^* \hat{F} \psi d\tau$

一维情况下：$\bar{F} = \int \psi^*(x)\hat{F}\psi(x)dx = \int (\Sigma_n c_n \phi_n)^* \hat{F} (\Sigma_m c_m \phi_m) dx$

$\because \hat{F}\phi_n = \lambda_n \phi_n$

$\Rightarrow \int (\Sigma_n c_n \phi_n)^* \hat{F} (\Sigma_m c_m \phi_m) dx = \int (\Sigma_n c_n \phi_n)^* \lambda_m \Sigma_m c_m \phi_m dx$

$= \Sigma_n \Sigma_m c_n c_m \lambda_m \int \phi_n^* \phi_m dx = \Sigma_n |c_n|^2 \cdot \lambda_n$

$\bar{F} = \Sigma_n |c_n|^2 \lambda_n$

其中$|c_n|^2$是本征值出现的几率；

当$|c_n|^2$是归一化的时候，λ_n是本征值；

当$|c_n|^2$非归一化时，$\frac{c_1^2}{c_1^2 + c_2^2 + \cdots + c_n^2} = \lambda_n$