# 第七课：算符性质

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1. 两个算符的关系：$\begin{cases}\text{对易}\\ \text{不对易}\end{cases}$
   a. 对易：$[\hat{o},\hat{\mu}]\equiv\hat{o}\hat{\mu}-\hat{\mu}\hat{o}=0\Rightarrow\hat{o}\hat{\mu}=\hat{\mu}\hat{o}$；
      1. 两个算符同时有确定值⇔ $\psi$是共同本征函数；
      2. 两算符有一组共同的完备的本征函数系⇔两算符对易；
      3. 一组力学量算符具有共同完备的本征函数系⇔它们两两对易；
   b. 不对易：设$\hat{F}\hat{G}-\hat{G}\hat{F}=ik=[\hat{F},\hat{G}]$称为对易关系；
2. 示例：证明测不准原理。

   引入$I(\xi)=\int\left|\xi\widehat{\Delta F}\psi-i\widehat{\Delta G}\psi\right|^2d\Omega\geq0$

   $$=\int\left(\xi\widehat{\Delta F}\psi-i\widehat{\Delta G}\psi\right)^*\left(\xi\widehat{\Delta F}\psi-i\widehat{\Delta G}\psi\right)d\Omega$$

   $$=\int\left(\xi^2(\Delta F\psi)^*\widehat{\Delta F}\psi-i\xi(\Delta\mathrm{F}\psi)^*\widehat{\Delta G}\psi\text{-}i\xi(\Delta G\psi)^*\Delta F\psi+\left(\widehat{\Delta G}\psi\right)^*\Delta G\psi\right)d\Omega$$

   $\because\Delta F=\hat{F}-\bar{F}$

   $(\Delta F)^+=\left(\hat{F}-\bar{F}\right)^+$

   $=\hat{F}^+-\bar{F}^+$

   $=\Delta F$

   $\Rightarrow\Delta F$是厄米算符

   $$=\xi^2\int\psi^*\widehat{\Delta F}^2\psi d\tau-i\xi\int\psi^*\left(\widehat{\Delta F}\Delta G-\Delta G\Delta F\right)\psi d\tau+\int\psi^*(\Delta G)^2\psi d\tau$$

   $\widehat{\Delta F}\Delta G-\Delta G\Delta F$

   $\equiv[\Delta F,\Delta G]$

   $=\left[\hat{F}-\bar{F},\hat{G}-\bar{G}\right]$

   $=\left[\hat{F},\hat{G}\right]-\left[\hat{F},\bar{G}\right]-\left[\bar{F},\hat{G}\right]+[\bar{F},\bar{G}]$

   $=\left[\hat{G},\hat{G}\right]$

   $=ik$

   $=\xi^2\overline{(\Delta F)^2}+\xi\bar{k}+\left(\widehat{\Delta\mathrm{G}}\right)^2\geq0$ (以ξ为变量的二次函数)

   $\because b^2-4ac\leq0$

   $\Rightarrow\bar{k}^2-4\overline{(\Delta F)^2}\cdot\overline{(\Delta\mathrm{G})^2}\leq0$

   $\Rightarrow\overline{(\Delta F)^2}\cdot\overline{(\Delta G)^2}\geq\dfrac{\bar{k}^2}{4}$

   当ΔF → Δx, ΔG → Δp:

   $[x,p_x]=i\hbar$

   $\Rightarrow\overline{(\Delta x)^2}\cdot\overline{(\Delta p)^2}\geq\dfrac{\hbar^2}{4}$

   证得：$\Delta x\cdot\Delta p\geq\frac{\hbar}{2}$，即测不准原理；
3. 示例：
   a. 证明：$\left[\widehat{L_x},\widehat{L_y}\right]=i\hbar L_z$

   $\left[y\widehat{P_z}-z\widehat{P_y},z\widehat{P_x}-x\widehat{P_z}\right]$

   $=\left[y\widehat{P_z},z\widehat{P_x}\right]-\left[y\widehat{P_z},x\widehat{P_z}\right]-\left[z\widehat{P_y},z\widehat{P_x}\right]+\left[z\widehat{P_y},x\widehat{P_z}\right]$

   $=\left[y\widehat{P_z},z\widehat{P_x}\right]+\left[z\widehat{P_y},x\widehat{P_z}\right]$

   $=y\left[P_z,z\widehat{P_x}\right]+\left[y,z\widehat{P_x}\right]\widehat{P_z}+z\left[\widehat{P_y},x\widehat{P_z}\right]+\left[z,x\widehat{P_z}\right]\widehat{P_y}$

   $=y\left[\widehat{P_z},z\widehat{P_x}\right]+\left[z,x\widehat{P_z}\right]\widehat{P_y}$

   $=yz\left[\widehat{P_z},\widehat{P_x}\right]+y\left[\widehat{P_z},z\right]\widehat{P_x}+x\left[z,\widehat{P_z}\right]\widehat{P_y}+[z,x]\widehat{P_y}\widehat{P_z}$

   $\because[P_z,X_z]=-i\hbar$

   $\Rightarrow$ 原式 $=y(-i\hbar)\widehat{P_x}+x\cdot i\hbar\widehat{P_y}$

   $=i\hbar\left(x\widehat{P_y}-y\widehat{P_x}\right)$

$= i\hbar\widehat{L_z}$

b. 证明：$L_z$的本征态$Y_{lm}$下，$<L_x>=<L_y>=0$

$[L_y, L_z] = i\hbar L_x \Rightarrow \widehat{L_x} = \frac{L_y \cdot L_z - L_z \cdot L_y}{i\hbar}$

$\because L_z Y_{lm} = m\hbar Y_{lm}$

$<L_x>=\frac{1}{i\hbar}\int Y_{lm}^*(L_y \cdot L_z - L_z \cdot L_y) \cdot Y_{lm} d\Omega$

$= \frac{1}{i\hbar}\int (Y_{lm}^* L_y L_z Y_{lm} - Y_{lm}^* L_z L_y Y_{lm}) d\Omega$

$= \frac{1}{i\hbar}\int (Y_{lm}^* L_y (m\hbar) Y_{lm} - (L_z Y_{lm})^* L_y Y_{lm}) d\Omega$

$= \frac{1}{i\hbar}\int (Y_{lm}^* L_y (m\hbar) Y_{lm} - (m\hbar) Y_{lm}^* L_y Y_{lm}) d\Omega$

$= 0$

c. 用测不准原理证明$L_z$在$Y_{lm}$下，$<L_x>=<L_y>=0$

$\because \overline{(\Delta L_y)^2} \cdot \overline{(\Delta L_z)^2} \geq \frac{\hbar^2}{4}\overline{L_x}^2$，　　$L_z$在$Y_{lm}$下$\overline{\Delta L_z}^2 = 0$

$\Rightarrow \frac{\hbar^2}{4}\overline{L_x}^2 \leq 0$

$\Rightarrow \overline{L_x}^2 \leq 0$

$\Rightarrow \overline{L_x}^2 = 0$

$\Rightarrow \overline{L_x} = 0$

$<L_x>=0$

同理$<L_y>=0$

d. 在$\hat{L}^2$和$\widehat{L_z}$共同本征态$Y_{lm}$下，求$\overline{(\Delta L_z)^2}$和$\overline{(\Delta L_y)^2}$

$\overline{\Delta F}^2$

$= \overline{(\hat{F} - \bar{F})^2}$

$= \overline{\hat{F}^2 - 2\hat{F} \cdot \bar{F} + \bar{F}^2}$

$= \overline{F^2} - 2\bar{\hat{F}} \cdot \bar{F} + \bar{F}^2$

$= \overline{F^2} - \bar{F}^2$

$\overline{\Delta L_x}^2 = \overline{L_x^2} - \overline{L_x}^2$

$\because \overline{L_x} = \overline{L_y} = 0$

$\Rightarrow \overline{\Delta L_x}^2 = \overline{L_x^2} = \int Y_{lm}^* L_x^2 Y_{lm} d\Omega$

$i\hbar L_x = [L_y, L_z]$

$= L_y \cdot L_z - L_z \cdot L_y$

$\Rightarrow i\hbar L_x \cdot L_x = L_y L_z \cdot L_x - L_z L_y \cdot L_x$

$\because L_z Y_{lm} = m\hbar Y_{lm}$

$\Rightarrow \quad = \frac{(L_y(L_x L_z + i\hbar L_y) - L_z L_y L_x)}{i\hbar}$

$\because [L_z, L_z] = i\hbar L_y$

$\Rightarrow \quad = \frac{1}{i\hbar}\int Y_{lm}^*(L_y L_x L_z + i\hbar L_y^2 - L_z L_y L_x) Y_{lm} d\Omega$

$= \frac{1}{i\hbar}\int Y_{lm}^*(m\hbar L_y L_x + i\hbar L_y^2 - m\hbar L_y L_x) Y_{lm} d\Omega$

$= \frac{1}{i\hbar}\int Y_{lm}^*(i\hbar L_y^2) Y_{lm} d\Omega$

$= \overline{L_y^2}$

$L_x^2 + L_y^2 + L_z^2 = L^2$

$= + L_x^2 + L_y^2 = L^2 - L_z^2$

在$Y_{lm}$下取平均:

$\int Y_{lm}^*(L_x^* + L_y^*) Y_{lm} d\Omega = \int Y_{lm}^*(L^2 - L_z^2) Y_{lm} d\Omega$

$\overline{L_x^2} + \overline{L_y^2} = l(l+1)\hbar^2 - (m\hbar)^2$

$\overline{L_x^2} = \overline{L_y^2} = \frac{[l(l+1) - m^2]\hbar^2}{2}$